TRANSFORMATIONS OF EINSTEIN SPACES

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In a paper which is to be published elsewhere are obtained all Einstein manifolds whose line elements are determined by a quadratic differential form of the type

$$ds^2 = f(xyzt)(dx^2 + dy^2 + dz^2) + g(xyzt)dt^2 \qquad (1)$$

where f is really a function of t. Of the ten apparently distinct solutions of the cosmological equations for an element of this type one represents a hypersphere,[1] two are characterized by the fact that f and g involve an essentially complex argument and in the remaining seven, f is in each case related to a Weierstrass $\wp$-function. This suggests that only three of the ten solutions are distinct and that the relations between the various solutions of each group are to be found by transformation of coördinates. It is the purpose of this note to develop a theorem which will establish these relations.

Because of the covariant character of the cosmological equations, any transformation

$$x = x(XYZT), \quad y = y(XYZT), \quad z = z(XYZT), \quad t = t(XYZT)$$

which sends the line element (1) into one of the same form

$$dS^2 = F(XYZT)\ (dX^2 + dY^2 + dZ^2) + G(XYZT)dT^2$$

necessarily sends any one of the solutions mentioned above into another solution—in some cases into itself. This group of transformations is of course that under which the differential equations for the determination of f and g are invariant. Since the 3 spaces determined by t = constant are conformal Euclidean, then, in particular, any transformation which sends the Euclidean line element

$$d\sigma^2 = dx^2 + dy^2 + dz^2$$

into a conformal Euclidean element

$$d\sigma^2 = \Lambda^2(XYZ)d\Sigma^2 \qquad d\Sigma^2 = dX^2 + dY^2 + dZ^2$$

must be of the desired form. This leads to the

THEOREM: If

$$x = x(XYZ), \qquad y = y(XYZ), \qquad z = z(XYZ)$$

is a transformation such that

$$d\sigma^2 = \Lambda^2(XYZ)d\Sigma^2$$

and $f(xyzt)$ and $g(xyzt)$ of (1) satisfy Einstein's cosmological equations, then

$$\begin{aligned} F(XYZt) &= \bar{f}(xyzt)\ \Lambda^2(XYZ) \\ G(XYZt) &= \bar{g}(xyzt) \end{aligned}$$

also define a solution. In $\bar{f}(xyzt)$ and $\bar{g}(xyzt)$ x, y and z are considered as functions of X, Y and Z.

Transformations of Kelvin's type[2]

$$x = \frac{aX}{Y+iZ} \quad y = \frac{R^2-a^2}{2(Y+iZ)} \quad z = \frac{R^2+a^2}{2i(Y+iZ)} \quad (R^2 = X^2+Y^2+Z^2)$$

satisfy the conditions of the above theorem, where

$$\Lambda(XYZ) = \frac{a}{Y+iZ}$$

Inversion with respect to a sphere is a less general transformation which may be obtained by two successive applications of those of Kelvin's type.

On applying the theorem developed above to the various line elements of form (1) it is found that but three are distinct. Letting

$$f = \frac{1}{\rho^2} \qquad g = \tau^2$$

they are

(1) A hypersphere;

(2) One in which

$$\rho = \rho(\xi, t) \qquad (\xi = y + iz)$$

is determined by any equation of the form

$$\frac{\partial^2\rho}{\partial\xi^2} = \rho^2(\xi)$$

where

$$\tau = \sqrt{\frac{3}{\lambda}}\frac{\partial}{\partial t}\log\rho,$$

λ is here the so-called cosmological constant, hence this solution degenerates if the cosmological equations are replaced by Einstein's original ones in which $\lambda = 0$;

(3) A spherically symmetric one from which seven of the line elements may be obtained by means of the theorem. In it

$$\rho = \rho(r, t) \qquad (r^2 = x^2 + y^2 + z^2)$$

where

$$\left(\frac{\partial \rho}{\partial r}\right)^2 - 2\frac{\rho}{r}\frac{\partial \rho}{\partial r} - 4\frac{\rho^3}{r^3} = D(t)$$

$$\tau = \frac{1}{\sqrt{\frac{\lambda}{3} - D(t)}}\frac{\partial}{\partial t}\log \rho$$

$D(t)$ being arbitrary. ρ may here be expressed in terms of a $\wp$-function whose argument is log r.

The theorem which is given above thus reduces the ten elements of form (1) to three fundamental ones. Corresponding theorems are easily developed for use in connection with line elements other than (1), e.g., for those of Kasner's type.[3]

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[1] Cf. Kasner, E., *Amer. J. Math.*, **43**, 20–28 (1921); *Math. Ann.*, **85**, 234–236 (1922).

[2] Bateman, H., *Electrical and Optical Wave Motion* (Cambridge, 1915), p. 31.

[3] Kasner, E., these PROCEEDINGS, **11**, 95–96 (1925).

ON THE EQUI-PROJECTIVE GEOMETRY OF PATHS

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1. *Introduction.*—In this note I wish to give an indication of the method of development of the equi-projective geometry of paths,[1] which, although quite analogous to that of the affine geometry, requires some special consideration. The theorems which I have given have their counterpart in the affine geometry of paths; yet the theorem in the last paragraph on complete sets of identities is new even for the affine geometry.[2]

2. *Characterization of Equi-projective Normal Coördinates.*—Equi-projective normal coördinates (z) are completely characterized by the equations

$$\mathfrak{P}^i_{\alpha\beta}\, z^\alpha z^\beta = 0 \qquad (2.1)$$

in terms of an affine connection $\mathfrak{P}^i_{\alpha\beta}$ which is obtained from the projective connection $\Pi^i_{\alpha\beta}$ in the general (x) coördinate system according to the equations

$$\mathfrak{P}^i_{\alpha\beta} = \frac{\partial z^i}{\partial x^\sigma}\left(\frac{\partial^2 x^\sigma}{\partial z^\alpha \partial z^\beta} + \Pi^\sigma_{\mu\nu}\frac{\partial x^\mu}{\partial z^\alpha}\frac{\partial x^\nu}{\partial z^\beta}\right) \qquad (2.2)$$